The Ultimate Flat Tire

Stan Wagon
Macalester College
St. Paul, MN 55105

wagon@macalester.edu

A square wheel rolls smoothly on a road made of linked catenaries. This article explains the mathematics behind this construction and describes a full-sized model that the public can ride.

■ The Flattest Wheel

How flat can a tire be? Well, it's hard to beat a straight line! Can a straight line be used as a wheel? Sure, if one uses some care in defining its center. This insight is due to G. B. Robison in 1960 [R]; he also realized that by suitably truncating a doubly infinite straight line one could form a square wheel, which would indeed roll on a properly shaped road.

To set the stage we must be precise about what "roll" means. A round circle rolls on a straight line in the sense that the center of the wheel stays horizontal. So for a noncircular wheel, we will say that it rolls on a curvy road if the center of the wheel moves in a horizontal line as the wheel moves without slipping along the road. The goal of this article is to explain why a square rolls on a sequence of inverted catenaries (recall that a *catenary* is the curve made by a flexible chain allowed to hang with both ends held at the same height; its equation is simply $y = \cosh x = (e^x + e^{-x})/2$).

■ As the Wheel Turns

Suppose the road is given as a function $y = f(x)$. Then the situation is summarized in figure 1, where $f(x)$ is $\cos(x) - \sqrt{17}$. (Exercise: After reading the analysis of this article, explain the significance of the $\sqrt{17}$ in this example.) As the wheel rolls, the distance from its center (0, 0) to the road must match match the depth of the road: this means that the two dashed lines have equal length. And the road surface and wheel circumference must match, so the two thick curves must have equal arc length. These conditions will allow us to get the polar equation $r = r(\theta)$ of the wheel suitable for the given road. The graph on the right shows the key function $\theta(x)$, which tells us the polar angle of the straight-down radius when the wheel has rolled enough so that its center is above the point x on the x-axis. We are using standard polar coordinates, so $\theta(0) = -\pi/2$.

■ Code for Figure 1.

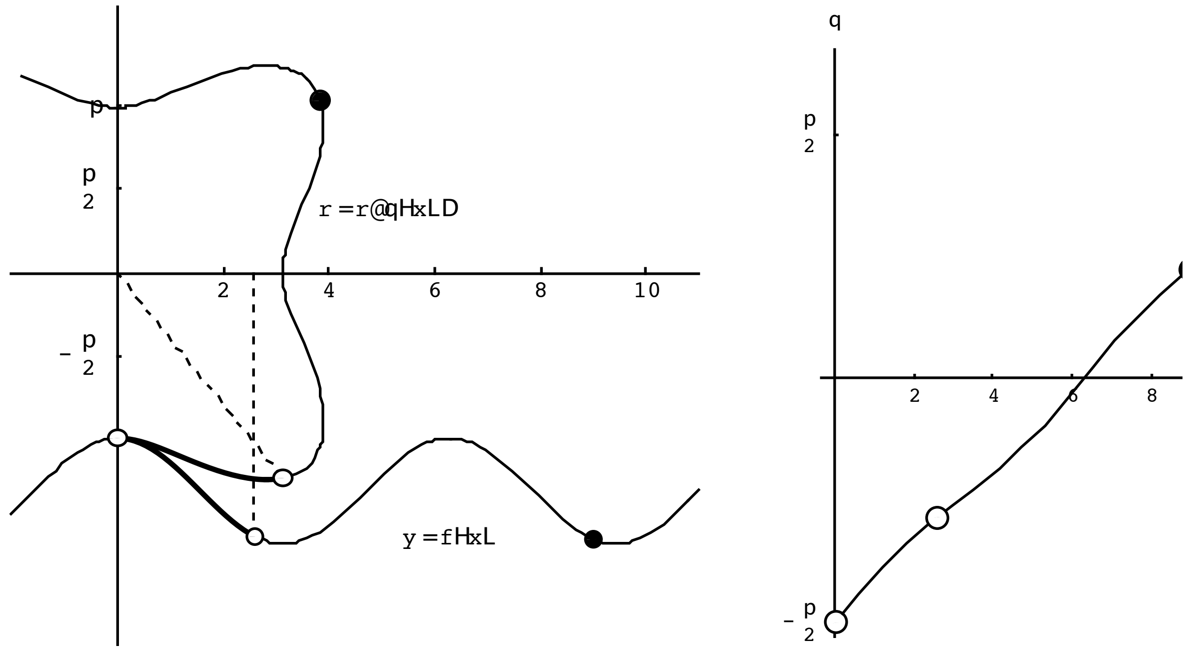

Figure 1. The left diagram shows a wheel about to begin rolling on a cosine-shaped road. The two dashed lines must have equal length, and the two thick curves must also have equal length. The image on the right shows the θ vs. x relationship.

Now, the condition corresponding to the dashed lines leads to the radius condition

$$r[\theta(x)] = -f(x) \tag{1}$$

The negative sign is included because r should be positive but f(x), which defines the road, will be negative. Note the initial condition becomes $r[\theta(0)] = -f(0)$, or $r(-\pi/2) = -f(0)$.

Next we match the arc lengths. The road length is given by the familiar formula, $\int_0^x \sqrt{1 + [f'(t)]^2}\, dt$, while the wheel circumference is the slightly less familiar

$$\int_{-\pi/2}^{\theta(x)} \sqrt{[r(\theta)]^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Equating these integrals, differentiating both sides with respect to x, and squaring yields:

$$1 + [f'(x)]^2 = \left(\frac{d\theta}{dx}\right)^2 \left[r(\theta)^2 + \left(\frac{dr}{d\theta}\right)^2\right] \tag{2}$$

But the radius condition (1) can be differentiated with respect to x to yield $(dr/d\theta)(d\theta/dx) = -f'(x)$. Substituting into (2) yields:

$$1 + [f'(x)]^2 = \left(\frac{d\theta}{dx}\right)^2 [r(\theta)^2] + [f'(x)]^2$$

which simplifies to

$$\frac{d\theta}{dx} = \frac{1}{r(\theta)}$$

This is what we want: a differential equation, quite simple as it turns out, that relates the rolling function $\theta(x)$ and the shape of the wheel $r(\theta)$. The variables separate into $r(\theta)\, d\theta = dx$ so integration, and the initial conditions, can be used to get x in terms of θ as follows.

$$x = \int_{-\pi/2}^{\theta} r(\theta)\, d\theta \tag{3}$$

If we can invert this to get a formula for $\theta(x)$, we will know the shape of the road, since, for any x, $f(x) = -r[\theta(x)]$.

Exercise: Show that if we match tangent slopes instead of matching arc lengths we get the same fundamental relationship (3), in a way that avoids the arc length integrals.

■ Why a Catenary

Now consider the straight wheel. If we take the origin as the center of this wheel, then its polar equation is $r = -\csc\theta$, $-\pi/2 < \theta < 0$.

■ Code for Figure 2

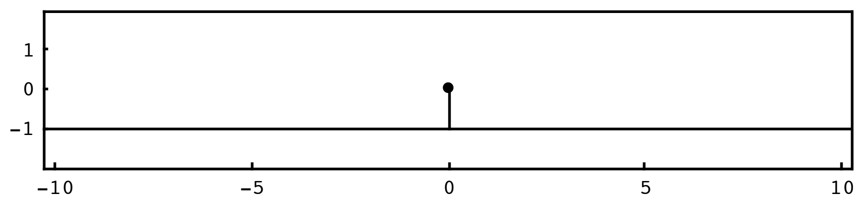

Figure 2. The polar equation $r = -\csc\theta$ has a straight line as its graph.

So to get the rolling relationship we evaluate the integral of (3)

$$x = \int -\frac{1}{\sin\theta}\, d\theta \tag{4}$$

Standard calculus techniques, using the facts that the θ-domain is $(-\pi, 0)$ and the initial condition is $x(-\pi/2) = 0$, tell us that

$$x = -\log\left(-\tan\frac{\theta}{2}\right)$$

This inverts to $\theta(x) = 2\arctan(-e^{-x})$. It follows that the road we seek is the graph of

$$y = r[\theta(x)] = -\csc[2\arctan(-e^{-x})]$$

This simplifies if we use the rule $\sin(2x) = 2(\sin x)(\cos x)$ in its reciprocal form, as we do with the substitution that follows, letting *Mathematica* do the algebra and trig. and multiplying by -1 to get f(x).

```
Expand[ - ( -Csc[2 ArcTan[-E^-x]]) /. Csc[2 a_] :> Csc[a] Sec[a] / 2 ]
```

$$-\frac{e^{-x}}{2} - \frac{e^{x}}{2}$$

This last is just $-(e^{x} + e^{-x})/2$, which is the familiar catenary arch, $-\cosh x$. So our straight line will roll on a catenary, as shown in figure 3.

■ Code for Figure 3

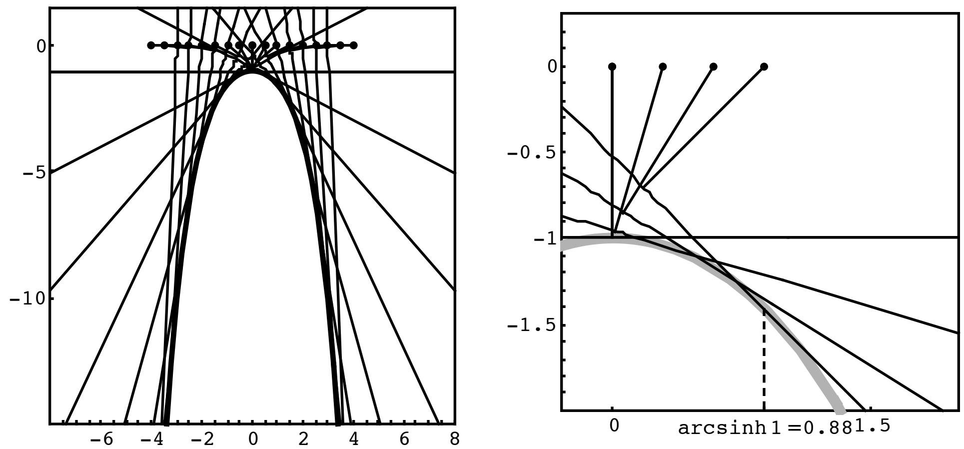

Figure 3. A wheel that is a straight line rolls on a catenary so that the hub, (0, 0), stays horizontal. The closeup view shows that when the center is at arcsinh(1), the line makes a 45° angle with the horizontal.

■ A Square Wheel

Now that we understand why the straight line rolls on the catenary it is easy to see how to handle a square wheel. Just truncate the catenary at the point at which the rolling straight line will make a 45° angle with the horizontal. (Exercise: What point is this?) Then when an identical truncated catenary is placed beside the first, the cusp will have a 90° angle; when a second straight line is placed perpendicular to the first one, we will have a rolling right angle. Do the same for the other angles of the square and, presto, a square wheel. Figure 4 shows how the wheel rolls; note that the locus of one of the corners occasionally goes backward, reminiscent of the classic puzzle about the locus of the point at the bottom of the flanged wheel of a train.

■ Code for Figure 4

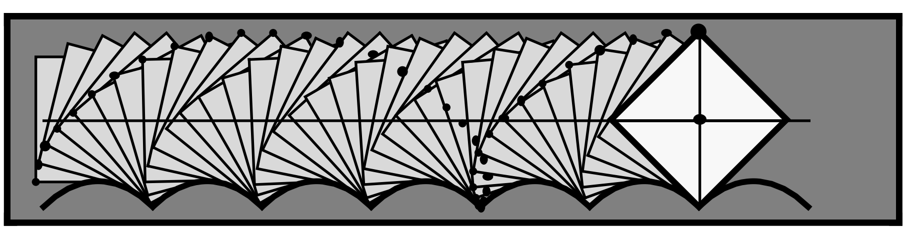

Figure 4. When a square rolls on a sequence of appropriately truncated catenaries, the ride is smooth in the sense that the hub of the wheels experiences no up or down motion. The dots show the locus of a corner.

Many people, on seeing the rolling square, wonder if a rolling pentagon or hexagon is possible. Indeed, essentially the same argument shows that any regular n-gon will roll on a catenary road. However, the case of a triangle leads to a certain phenomenon that renders a working model impossible to build. Exercise: Why is a triangular wheel especially difficult?

■ Is the Ride Smooth?

There is a subtle difference between a square wheel and a round one. For a normal bike, x is proportional to θ, where x is the distance traveled and θ is the angle pedaled. If you pedal faster, you travel farther, and the correspondence is linear:

pedal twice as fast and you travel twice as fast. This is *almost* true of the square wheel, but not quite. Figure 5 shows this relationship for the square wheel, with a straight line shown for comparison. The discrepancy is so small that it cannot be felt by a rider. Exercise: Examine the Maclaurin series of θ(x), which is 2 arctan(-e^{-x}), to see more clearly how it differs from linearity.

■ Code for Figure 5

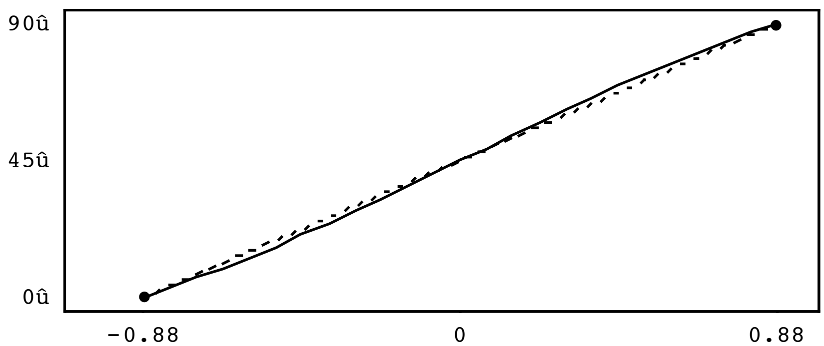

Figure 5. The θ vs. x relationship for a square wheel. The graph is very close to a straight line (dashed) so that even though horizontal speed is not a linear function of pedaling speed, the difference is very small, and not detectable on a full-size square-wheel bicycle.

■ A Working Model

Inspired by various models I had seen (a small one at San Francisco's Exploratorium and a larger one built by the Center of Science and Industry in Columbus, Ohio, I asked Loren Kellen, a neighbor who knows carpentry and bicycles, if he could build a working model of a square-wheel bike. He was enthusiastic and six months later it was done; the full-size model (the road is 23 feet long) is on permanent display in the science center at Macalester and is open for public riding. We decided on a three-wheel design for stability. Friction is a big concern: there must be enough friction between the tire and the road to prevent slippage, or "creep" as it is called by professionals in catenary road-building. Also the bike frame had to be sawn in two and rewelded so that the frame would fit the road, whose spacing is in turn decided by the size of the square wheels. I thought steering would be a problem, but in fact one can steer the thing provided one does so at the top of each arch!

■ More Surprises

Thanks to the power of modern software (*Mathematica*) an investigation such as this often leads to new insights. Leon Hall and I, after seeing the Exploratorium model, made an extensive investigation into the shapes of various road–wheel pairs. One surprising discovery is related to the age-old definition of a cycloid as the locus of a point on a round wheel rolling on a straight road. We found that the locus of a point of a limaçon as it rolls smoothly on a trochoid (itself a type of cycloid) is also an exact cycloid! Thus the cycloid we know and love can be viewed as one of a matched pair (see Figure 6). See [HW] for more such relationships. Here is a final puzzle, due to Robison: Find the unique road–wheel pair for which the road and wheel have *identical* shapes.

■ Code for Figure 6 (takes a minute or so)

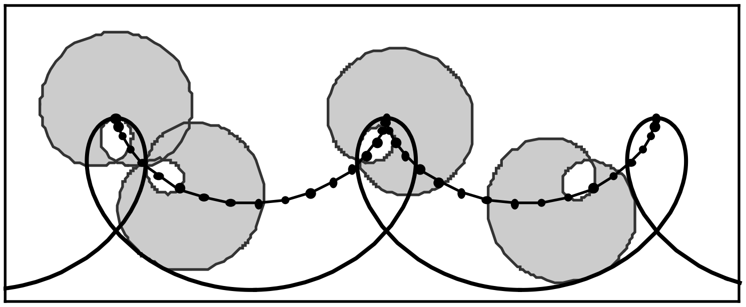

Figure 6. When a limaçon rolls on a trochoid the locus of one of the wheel's points is an exact cycloid.

■ References

[HW] L. Hall and S. Wagon, Roads and wheels, *Mathematics Magazine* 65 (1992) 283–301.

[R] G. B. Robison, Rockers and rollers, *Mathematics Magazine* 33 (1960) 139–144.

Editor: The solution to the final puzzle, FYI, is that road must be the parabola $y = -x^2 - 1/4$. Wheel is then $x^2 - 1/4$.