A Paradox Arising from the Elimination of a Paradox

Abstract. We present a result of Mycielski and Sierpiński—remarkable and underappreciated in our view—showing that the natural way of eliminating the Banach–Tarski paradox by assuming all sets of reals to be Lebesgue measurable leads to another paradox about division of sets that is just as unsettling as the paradox being eliminated. The *division paradox* asserts that the reals can be divided into nonempty classes so that there are strictly more classes than there are reals.

1. INTRODUCTION. The Zermelo–Fraenkel axioms (ZF) were introduced a century ago to avoid logical paradoxes, notably Bertrand Russell's: Does the set consisting of every set that is not a member of itself contain itself? The axiom of choice (AC)—first stated by Zermelo in 1904—asserts that for every set M of nonempty sets, there exists a set consisting of exactly one element from each set in M. When AC is added, the resulting system is called ZFC. Excellent books about the historical and technical aspects of AC are [**7**, **9**, **12**].

Working mathematicians don't construct formal proofs in ZFC or any other system. Rather, they construct arguments that their peers find convincing. Nevertheless, these arguments almost always correspond to formal derivations in ZF or ZFC. In short, while alternatives have been investigated, ZFC has proven to be a robust and adequate foundation for modern mathematics.

Yet some mathematicians harbor a nagging fear that AC might be too powerful. The issue is not that AC might yield an inconsistency—Gödel proved that it does not—but rather that it leads to inconveniences such as nonmeasurable sets and, more to the point, strikingly counterintuitive results. A recent example is in [**5**, **6**], but really nothing underscores the point more than the Banach–Tarski paradox, which shows that a solid ball B in $\mathbb{R}^3$ decomposes into five pairwise disjoint sets A_i so that $B = \rho_1 A_1 \cup \rho_2 A_2 = \rho_3 A_3 \cup \rho_4 A_4 \cup \rho_5 A_5$, where the unions are disjoint and ρ_i are rigid motions of $\mathbb{R}^3$ (see [**26**]). This duplication of a ball defies all reason, at least for those who are uncomfortable with the concept of a nonmeasurable set. And the culprit is indeed AC.

For example, noted physicist and mathematician Roger Penrose wrote [**17**, pp. 14–15]:

> Most mathematicians would probably regard the axiom of choice as 'obviously true', while others may regard it as a somewhat questionable assertion which might even be false (and I am myself inclined, to some extent, towards this second viewpoint).

He goes on to say (p. 366):

> It's not altogether uncontroversial that the axiom of choice should be accepted as something that is universally valid. My own position is to be cautious about it. The trouble with this axiom is that it is a pure 'existence' assertion, without any hint of a rule whereby the set might be specified. In fact, it has a number of alarming consequences. One of these is the Banach–Tarski theorem.

The most natural way to eliminate the paradox is to abandon AC and adopt the axiom that all sets are Lebesgue measurable. Under this addition, the Banach–Tarski paradox evaporates (see note 1 in Section 8). Our purpose here is to present and prove results of Mycielski and Sierpiński that are not generally known and show that this option, which sounds so reasonable, actually has a serious drawback that leads to another paradox, one that is just as disconcerting as the one being eliminated. Since the use of AC has not proven to be a problem for mathematics—we have learned to live with nonmeasurable sets—this underscores the accepted view that ZFC is the proper foundation for mathematics.

To set the stage, consider the National Football League, with its 32 teams and $32 \cdot 53 = 1696$ players. If the players were assigned to teams in some other way, subject only to the conditions that a team cannot have zero players and each player can be on only one team, then there can certainly be more than 32 teams. But could there be more than 1696 teams? Of course not. The idea of grouping players into nonempty teams so that the teams outnumber the players is ludicrous. It's like finding a country that has more populated provinces than it has people. Yet this is the essence of the phenomenon that we will present: this sort of thing can arise in a mathematical world without AC.

The main assertion we study here is analogous to the sports team example. The players are the real numbers with two reals placed on the same team if and only if they differ by a rational. That is, we will look at $\mathbb{R}/\mathbb{Q}$, the quotient group of the additive group of reals using the subgroup of rationals. Consider the statement that $|\mathbb{R}| < |\mathbb{R}/\mathbb{Q}|$ (where $|\cdot|$ is cardinality; see Section 2). This says that there are strictly more equivalence classes of reals than there are reals; we call this assertion the *division paradox*.

2. PRELIMINARIES. The $\mathbb{R}/\mathbb{Q}$ equivalence relation has $x \sim y$ if and only if $x - y \in \mathbb{Q}$; we use $[x]$ for the equivalence class $x + \mathbb{Q}$ of a real x. Vitali used this relation to construct the first nonmeasurable set. He used AC to get X containing exactly one real from each class in $\mathbb{R}/\mathbb{Q}$; then X is not Lebesgue measurable. Our interest is in sets that are somewhat the opposite of what Vitali considered. A set A of reals is *Q-invariant* if $A = A + q$ for every rational q; such a set is a union of some classes in $\mathbb{R}/\mathbb{Q}$.

Cardinality is denoted by $|\cdot|$; in a choice-challenged world it is defined by: $|X| = |Y|$ if there is a bijection from X to Y. By the classic Schröder–Bernstein theorem, a theorem of ZF, this is equivalent to $|X| \le |Y|$ and $|Y| \le |X|$, where $|A| \le |B|$ means that there is a one-one function $f : A \to B$. Further, $|X| < |Y|$ means that $|X| \le |Y|$ and $|X| \ne |Y|$. Under the axiom of choice, every set has a cardinality from the well-ordered collection $0, 1, 2, \ldots, \aleph_0, \aleph_1, \aleph_2, \ldots$; without AC, there exist incomparable sets: X and Y such neither $|X| \le |Y|$ nor $|Y| \le |X|$ holds.

A subset of $\mathbb{R}$ is *open* if it is the union of open intervals; it is *nowhere dense* if every nonempty open interval contains a nonempty subinterval disjoint from it. A *meager set* is a countable union of nowhere dense sets; a *comeager* set is one whose complement is meager. A set A of reals has the *property of Baire* if A differs from some open set G by a meager set M (meaning $A = G \triangle M$, where $\triangle$ is symmetric difference).

Every nested sequence of nonempty, bounded, closed sets has a nonempty intersection; this is Cantor's intersection theorem, a theorem of ZF. The case of closed intervals is easy: the least upper bound of the left endpoints is in the intersection. As a consequence, the real line is not meager; given nowhere dense sets N_i, construct a nested sequence of closed intervals with rational endpoints so that the nth interval is disjoint from N_n and then intersect them all. The use of rational endpoints means that this does not use the axiom of choice. **Alan: This might be one place where the referee wondered why AC is not used. I guess it is like this: For each i, let f(i) be the least j such that there is a k so that the rational interval $[q_j, q_k]$ is disjoint from N_i, using a fixed enumeration of the rats. Then lub $q_{f(i)}$ is a real not in the union. But do we really want to say this? This can always be phrased as a question to the editor.... Note that we do not need that $\mathbb{R}$ is nonmeager until §5, but we do use it there.**

And note that this ¶ is about Cantor intersection for closed sets of reals. Later in the paper we use Cantor intersection for compact sets in a Hausdorff space.

Suppose now that A is a $\mathbb{Q}$-invariant set of reals that is measurable or has the Baire property. Let λ denote Lebesgue measure. Remarkably, in the measure case, if J is any interval, $\lambda(A \cap J)$ is either 0 or $\lambda(J)$, and in the Baire case either A or $\mathbb{R} \setminus A$ is meager (we prove both in a moment). We need the fact that any Lebesgue measurable A is contained in a union of open intervals with rational endpoints for which the sum of all the interval lengths is arbitrarily close to $\lambda(A)$. This is true because $\lambda(A)$ is defined to be the outer measure of A, which is the greatest lower bound of the aforementioned sums for countable sets of intervals that cover A. For more on zero-one laws, see Section 5, and also [**16**, chapter 21].

Theorem 1 (Zero-one law for $\mathbb{R}/\mathbb{Q}$; ZF). *Let $A \subseteq \mathbb{R}$ be $\mathbb{Q}$-invariant. If A has the Baire property, then A is either meager or comeager. If A is measurable, then either (a) A intersects all bounded intervals in measure zero; or (b) A intersects all bounded intervals J in measure $\lambda(J)$. If λ is restricted to* $[0,1]$*, then any set that is $\mathbb{Q}$-invariant (modulo* 1*) has measure* 0 *or* 1.

Proof. For the first, assume A is nonmeager and M and G witness A having the Baire property; then $G \ne \emptyset$. We claim that if $y \notin A$, then $y \in \bigcup_{q \in \mathbb{Q}} M + q$, which proves that A is comeager. To prove the claim, choose $q \in \mathbb{Q}$ so that $y \in G + q$. This is possible because the rational translates of any nonempty interval cover $\mathbb{R}$. Thus $y - q \in G$. Since $y \notin A$ and A is $\mathbb{Q}$-invariant, we have $y - q \notin A$; so $y - q \in G \setminus A \subseteq M$. Hence $y \in M + q$.

Now suppose A is Lebesgue measurable and $\mathbb{Q}$-invariant; let $B = A \cap [0,1]$ and $\alpha = \lambda(B)$. We will show that $\lambda(A \cap J) = \alpha\, \lambda(J)$ for any interval J with rational endpoints. Letting $m, n \in \mathbb{N}$, $\mathbb{Q}$-invariance implies that this holds for $[m, m+1]$; it then extends to $[0, m]$ by subdividing into unit intervals. Division of $[0, m]$ into n equal subintervals then gives the property for $[0, m/n]$, from which one gets it for all intervals with rational ends having length m/n. Now suppose $0 < \alpha < 1$. There is a family of intervals with rational endpoints $\{K_i\}_{i=0}^{\infty}$ covering B and having $\sum_{i=0}^{\infty} \lambda(K_i) = \beta$, with $\alpha \le \beta < 1$; let $\epsilon = \alpha\,(1 - \beta)$. The tail of the series approaches 0, so there is a finite union $\bigcup_{i=0}^{n} K_i$ that covers B except for a set of measure less than ϵ. Finite subadditivity of λ then gives the following contradiction

$$\alpha = \lambda(B) < \epsilon + \lambda\left(\bigcup_{i \le n} B \cap K_i\right) \le \epsilon + \sum_{i \le n} \lambda(B \cap K_i) = \epsilon + \alpha \sum_{i \le n} \lambda(K_i) \le \alpha(1-\beta) + \alpha\beta = \alpha.$$

The last sentence of Theorem 1 follows easily by considering $\bigcup\{A + n : n \in \mathbb{Z}\}$. ∎

When working without the axiom of choice, one usually replaces it with a weaker version known as the axiom of dependent choice (DC). Without something like this, λ might not be countably additive (worse, a countable union of countable sets might not be countable). The statement of DC is: If $*$ is a binary relation on a nonempty set X and for every $x \in X$ there is $y \in X$ with $x * y$, then there is a sequence $(x_n)_{n \in \mathbb{N}}$ such that $x_n * x_{n+1}$ for every $n \in \mathbb{N}$. Countable additivity of λ

(and the same for meager sets) follows from the axiom of choice for countable families of nonempty sets, a consequence of DC.

We use LM for the assertion that all sets of reals are Lebesgue measurable. The theory ZF + DC + LM is consistent, provided one assumes the consistency of the existence of an inaccessible cardinal (note 2). This is a remarkable connection, especially because the inaccessible is both necessary and sufficient for this [**18**, **19**, **24**]. Because it appears that inaccessibles do not introduce a contradiction, we will treat ZF + DC + LM as we do ZF: they are assumed to be consistent. Similarly, we use PB for the assertion that all sets of reals have the property of Baire. The theory ZF + DC + PB is equiconsistent with ZF [**19**, **25**]; the contrast to the connection of LM to large cardinals is surprising.

3. THE DIVISION PARADOX. We give here a self-contained and short proof of the division paradox in the context of the familiar additive group $\mathbb{R}$ and its rational subgroup. More precisely, we show that $|\mathbb{R}| < |\mathbb{R}/\mathbb{Q}|$ is a theorem of ZF (with no assumption of any form of AC) when either LM or PB is assumed. Theorem 2 is due to Mycielski [**13**, **15**] and Theorem 3 to Sierpiński [**20**, **21** (§8), **22**, **23**].

Theorem 2 (ZF). $|\mathbb{R}| \le |\mathbb{R}/\mathbb{Q}|$.

Theorem 3 (ZF). *If* $\mathbb{R}/\mathbb{Q}$ *has a linear ordering* $\preccurlyeq$, *then* $A = \{x \in \mathbb{R} : [x] \preccurlyeq [-x]\}$ *is* $\mathbb{Q}$*-invariant, is not Lebesgue measurable, and does not have the property of Baire.*

An injection from $\mathbb{R}/\mathbb{Q}$ to $\mathbb{R}$ induces a linear ordering on $\mathbb{R}/\mathbb{Q}$ and so these theorems immediately yield the following.

Corollary 4. *The division paradox* $|\mathbb{R}| < |\mathbb{R}/\mathbb{Q}|$ *is a theorem in either* ZF + LM *or* ZF + PB.

Under AC one can find a choice set V for the equivalence classes in $\mathbb{R}/\mathbb{Q}$; this yields $|\mathbb{R}/\mathbb{Q}| \le |\mathbb{R}|$. Being nonmeasurable, V cannot exist under LM. Theorem 3 is stronger: in ZF + (LM or PB), not only is there no choice set, but there is no injection of any sort from $\mathbb{R}/\mathbb{Q}$ into $\mathbb{R}$, nor any linear ordering of $\mathbb{R}/\mathbb{Q}$.

The proof of Theorem 3 given here, with its use of a linear order, is based on the ideas used by Sierpiński (alternate approaches are in Section 5). Under AC there is a well-ordering of the reals; that yields a choice set for the classes by choosing the least element in each class. So the well-ordering gives a nonmeasurable set. Theorem 3 shows that a linear ordering of the classes is enough to get a nonmeasurable set.

The proof of Theorem 2 that follows is a specialization of a more general approach (see Theorem 7).

Proof of Theorem 2. Because $x \mapsto \frac{1}{1-x} - \frac{1}{x}$ is a bijection of $(0,1)$ with $\mathbb{R}$, we need only inject $(0, 1)$ into $\mathbb{R}/\mathbb{Q}$. Enumerate $\mathbb{Q}$ as $\{q_m\}_{m\ge1}$. Build a tree by taking $(0, 1)$ as the root. For level n, first shrink each interval at level $n-1$ to a size smaller than q_n and then choose two disjoint open subintervals of the shrunken interval. This ensures that no two points at level n differ by any q_m, $m \le n$, and also that the interval-lengths shrink to 0 along any branch. Given $x \in (0, 1)$, let $s \in 2^{\mathbb{N}}$ be its base-2 expansion, avoiding sequences with a tail of 1s. Taking 0 as left and 1 as right, s gives a branch in the tree and the Cantor intersection theorem yields a unique real y_x in all the intervals in the branch. The tree construction implies that the points y_x, $0 < x < 1$, are inequivalent, as required. ■ **XX It is not really necessary to call on C.I.Thm, but it is ok to do so.**

Proof of Theorem 3. Because $x \in \mathbb{Q}$ if and only if $2x \in \mathbb{Q}$, we have $x \sim -x$ if and only if $x \in \mathbb{Q}$. This means that $\mathbb{Q} \subseteq A$ and also that $\rho(x) = -x$ defines a bijection from the irrationals in A to $\mathbb{R} \setminus A$ that preserves measure and meagerness. Because $x \sim x + q$ and $-x \sim -(x + q)$, A is $\mathbb{Q}$-invariant. Assume A has the property of Baire. By the zero-one law, A is either meager or comeager. But A is meager if and only if $A \setminus \mathbb{Q}$ is meager if and only if $\rho(A \setminus \mathbb{Q})$ is meager if and only if $\mathbb{R} \setminus A$ is meager, contradiction. Similar reasoning, with meager replaced by measure 0 and working with $A \cap [-1, 1]$, works for the measure case. ■

To see why the division paradox is surprising, recall that cardinality is a partial order and so, in ZF, there are four possibilities for the cardinality relation between $\mathbb{R}$ and $\mathbb{R}/\mathbb{Q}$:

1. $|\mathbb{R}| = |\mathbb{R}/\mathbb{Q}|$
2. $\mathbb{R}$ and $\mathbb{R}/\mathbb{Q}$ are incomparable: $|\mathbb{R}/\mathbb{Q}| \not\le |\mathbb{R}|$ and $|\mathbb{R}| \not\le |\mathbb{R}/\mathbb{Q}|$
3. $|\mathbb{R}/\mathbb{Q}| < |\mathbb{R}|$
4. $|\mathbb{R}| < |\mathbb{R}/\mathbb{Q}|$

The first choice is viable because it follows from AC: a choice set for the classes implies $|\mathbb{R}/\mathbb{Q}| \le |\mathbb{R}|$; then Theorem 2 and the Schröder–Bernstein theorem give equality. At first glance, one might expect (2) and (3) to be consistent with ZF. If AC is false, there will be incomparable sets, so perhaps $\mathbb{R}$ and $\mathbb{R}/\mathbb{Q}$ could be such a pair; and because the continuum hypothesis can fail, there could be room beneath the continuum for an uncountable set such as $\mathbb{R}/\mathbb{Q}$, which would give (3). But Theorem 2 shows directly that (2) and (3) are always false. So (1)'s failure implies (4) and (4) says that the set that one expects to be smaller, $\mathbb{R}/\mathbb{Q}$, is in fact strictly larger. Corollary 4 shows that (4) is true in some situations.

A consequence of Corollary 4 is the interesting result that either LM or PB negates the generalized continuum hypothesis [**13**]. An injection from $\mathcal{P}(\mathbb{R}/\mathbb{Q})$ to $\mathcal{P}(\mathbb{R})$ is given by $A \mapsto \bigcup A$. An injection in the other direction starts by forming the surjection from $\mathbb{R}/\mathbb{Q}$ to $\mathbb{R}$ given by sending $[y_x]$ to x and all other classes to 0, where y_x is as in the proof of Theorem 2. By inverse images, this induces an injection from $\mathcal{P}(\mathbb{R})$ to $\mathcal{P}(\mathbb{R}/\mathbb{Q})$, and so the two power sets have the same cardinality. Corollary 4 then gives $|\mathbb{R}| < |\mathbb{R}/\mathbb{Q}| < |\mathcal{P}(\mathbb{R}/\mathbb{Q})| = |\mathcal{P}(\mathbb{R})|$, showing that GCH fails. This is not a surprise since it is known that $\neg$ AC $\Rightarrow \neg$ GCH, but is a concrete example of the failure.

Those who find Banach–Tarski duplications unpalatable but want to keep the many useful consequences of the axiom of choice can work in ZFC and try, whenever possible, to restrict themselves to measurable sets. A constructive point of view also elucidates the division paradox, which can be studied profitably in ZFC provided one reinterprets cardinality.

The key idea is *Borel cardinality*: a naive view would define $|X| \leq_B |Y|$ to mean that there is a one-one Borel function $F: X \to Y$. But there is no natural topology on $\mathbb{R}/\mathbb{Q}$ and so instead this definition is used: Suppose X and Y are complete separable metric spaces (known as *Polish spaces*), each endowed with an equivalence relation ($\sim$ denotes either one) and with the collections of equivalence classes denoted $\hat{X}$ and $\hat{Y}$; then $|\hat{X}| \leq_B |\hat{Y}|$ means that there is a Borel function $F: X \to Y$ such that $a \sim b \Leftrightarrow F(a) \sim F(b)$. The proof of Corollary 4, with no essential change, yields the nonparadoxical and interesting result that $|\mathbb{R}| <_B |\mathbb{R}/\mathbb{Q}|$ (where the equivalence relation on $\mathbb{R}$ is equality). That is, there is a nicely definable injection from $\mathbb{R}$ to $\mathbb{R}$ that respects the rational equivalence relation in the codomain, but no such injection that respects the rational relation in the domain. More precisely, $|\mathbb{R}/\mathbb{Q}| \not\leq_B |\mathbb{R}|$: there is no Borel function $F: \mathbb{R} \to \mathbb{R}$ such that $x - y \in \mathbb{Q} \Leftrightarrow F(x) = F(y)$. For the theory and applications of Borel cardinality and Borel equivalence relations see [**2**, **4**, **11**].

4. CURIOUSER AND CURIOUSER. We motivated the division paradox by imagining a sports league having more teams than players. We could equally well have phrased this in terms of more conferences than teams: the conferences divide up the teams just as the teams partition the players. But can there be more conferences than teams *and* more teams than players? We'll refer to any example of this as a double division paradox.

Definition. A *double division paradox* is a triple (X, Y, Z) such that $|X| < |Y| < |Z|$ with surjections $f: X \to Y$ and $g: Y \to Z$.

If (X, Y, Z) is a double division paradox, then we can think of X as the player pool, with x_1 and x_2 on the same team if $f(x_1) = f(x_2)$, and with teams y_1 and y_2 in the same conference if $g(y_1) = g(y_2)$. The argument after the proof of Theorem 3 easily extends to show that a double division paradox yields a double failure of the GCH: $|X| < |Y| < |Z| < |\mathcal{P}(X)|$.

As this section's title suggests, a double division paradox can exist (in the absence of AC). The following example (and the comment after Theorem 6) was provided by Asaf Karagila and is included with his permission; ω_1 is the smallest uncountable ordinal.

Theorem 5 (ZF). *If no uncountable set of reals can be well-ordered, then* $(\mathbb{R}, \mathbb{R} \bigcup \omega_1, \mathbb{R} \times \omega_1)$ *is a double division paradox starting with* $\mathbb{R}$.

Proof. We need Lebesgue's classic surjection from $\mathbb{R}$ to ω_1: identify $\mathbb{R}$ with $\mathcal{P}(\mathbb{N} \times \mathbb{N})$ (both have the cardinality of $2^{\mathbb{N}}$) and map each set of pairs that is a well-ordering to its order type, sending other sets to 0. There is a surjection from $\mathbb{R}$ to $\mathbb{R} \times \mathbb{R}$ (e.g., by the method in note 1) and using Lebesgue's map on the second coordinate turns it into a surjection from $\mathbb{R}$ to $\mathbb{R} \times \omega_1$. This induces the two surjections needed for the paradox.

Injections from $\mathbb{R}$ to $\mathbb{R} \bigcup \omega_1$ to $\mathbb{R} \times \omega_1$ are trivial, so it remains to show that there are no injections in the reverse direction. An injection of $\mathbb{R} \bigcup \omega_1$ into $\mathbb{R}$ maps ω_1 to an uncountable set of reals admitting a well-ordering, contrary to the theorem's assumption. Finally, suppose $f: \mathbb{R} \times \omega_1 \to \mathbb{R} \bigcup \omega_1$ is one-one. For real x, let $A_x = f(\{x\} \times \omega_1) \bigcap \omega_1$. If no A_x is empty, then $\mathbb{R}$ is well-ordered by the least ordinal in A_x, contradiction. If $A_x = \emptyset$, then f embeds $\{x\} \times \omega_1$ into $\mathbb{R}$, again a contradiction. ■

Theorem 6. *Under* ZF + DC + LM, *there is a double division paradox starting with* $\mathbb{R}$.

Proof. Shelah [**19**, Theorem 5.1B] showed that the hypothesis of Theorem 5 holds in ZF + DC + LM. ■

One can go farther. There can be a quadruple paradox: more teams than players, more conferences than teams, more leagues than conferences, and more sports than leagues. And the axiom of determinacy [**13**] implies that $(\mathbb{R}, \mathbb{R} \bigcup \omega_1, \mathbb{R} \bigcup \omega_2, \mathbb{R} \bigcup \omega_3, \ldots)$ is an infinite division paradox. **Alan: Small edit needed. The phrase "There can be a quadrauple paradox is too vague since "can be" doesn't say whether it follows from LM or simply is consistent. In submission we said that Coehn reals lead to it. Well, that is pretty irrelevant to the LM world. So we should delete**

that sentence, byt retain the AD one. In summary, we have the double coming from LM and infinite from AD. Those things are relevant. Agree?

5. ALTERNATE SETTINGS FOR THE PARADOX. The division paradox was first established not for $\mathbb{R}/\mathbb{Q}$ but for the quotient group arising from the group $(\mathcal{P}(\mathbb{N}), \triangle)$ and its subgroup $\mathcal{F}$ consisting of the finite subsets of $\mathbb{N}$. And there are other alternate settings: two important relations are the Bernoulli shift and the tail relation. For the tail relation the underlying set is $\mathcal{P}(\mathbb{N})$ (viewed as $2^{\mathbb{N}}$) with sequences s and t being *tail-equivalent* if there are $m, n \in \mathbb{N}$ so that $s_{m+k} = t_{n+k}$ for all $k \in \mathbb{N}$. The shift starts with the integer-indexed binary sequences $2^{\mathbb{Z}}$, with equivalence given by the *shift map*: $s \sim_{\text{shift}} t$ if and only if there is an integer k so that, for every n, $s_n = t_{n+k}$.

An attractive feature of $\sim_{\text{shift}}$ is that the equivalence class of an anchored sequence $\ldots xyz\,\boldsymbol{a}\,bcd\ldots$ in $2^{\mathbb{Z}}$ (the anchor—the 0-coordinate—is $\boldsymbol{a}$) is just the same object with the anchor omitted: $\ldots xyzabcd\ldots$; the no-origin version may be viewed as the set of all shifts of an anchored form of it. The division paradox for $2^{\mathbb{Z}}$ says that there are more anchored doubly infinite sequences than unanchored ones, a conclusion that is just as absurd as the paradox for $\mathbb{R}$ and $\mathbb{Q}$.

There are various approaches to the division paradox in these alternative settings. An elegant method is to generalize the proofs of Section 3 to apply to all four contexts at once. Most of what follows is part of the folklore that includes work of Sierpiński, Mycielski, and others, as well as more recent work in the area of Borel equivalence relations.

The topology on $2^{\mathbb{N}}$ is the usual product topology from the discrete set $\{0,1\}$ and the measure (denoted λ) on $2^{\mathbb{N}}$ is the product measure from $\{0,1\}$, where $\{0\}$ and $\{1\}$ each get measure $1/2$. The natural map $f: 2^{\mathbb{N}} \to [0,1]$ via binary expansions is not one-one (a rational of the form $m/2^n$ arises from a sequence ending in only 0s and another ending in 1s), but it does induce a bijection from $2^{\mathbb{N}} \setminus f^{-1}(D)$ to $[0,1] \setminus D$ where D is the set of rationals of the form $m/2^n$. Because countable sets are meager and have measure 0, this bijection allows one to show that LM and PB are equivalent to the corresponding assertions in $2^{\mathbb{N}}$. Topology and measure in $2^{\mathbb{Z}}$ are similar (e.g., a basic open set is the set of sequences extending a fixed finitely specified sequence and the measure of such an open set is defined to be 2^{-m} where m is the number of components specified; the standard outer measure construction then yields the product measure).

We start with a unified approach that yields Corollary 4 for $\mathbb{R}/\mathbb{Q}$, the finite set and tail relations on $2^{\mathbb{N}}$, and the Bernoulli shift. In what follows, $\sim$ is an equivalence relation on X, which is one of $\mathbb{R}$, $2^{\mathbb{N}}$, or $2^{\mathbb{Z}}$ (though the arguments work for any uncountable Polish space).

The next result generalizes Theorem 2.

Theorem 7 (ZF). *Suppose $\sim$ is meager in $X \times X$. Then $|X| \leq |X/\sim|$.*

Proof. The proof is similar to the proof of Theorem 2. Start with X and build a tree with level n consisting of small closed balls that are disjoint from the first n sets in the representation of the meager relation. Intersections along a branch are nonempty by the Cantor intersection theorem for compact sets. We leave the details to the reader. ■

The relations we are studying all satisfy the hypothesis of Theorem 7. For the tail relation define, for each $j, k \in \mathbb{N}$, $N_{j,k}$ to be $\{(s,t) : s \text{ beyond } j \text{ equals } t \text{ beyond } k\}$. Then the tail relation is $\bigcup N_{j,k}$ and each $N_{j,k}$ is nowhere dense in $2^{\mathbb{N}} \times 2^{\mathbb{N}}$, as we show next. Working with binary sequences, an open set in the product contains the product of two basic open sets in $2^{\mathbb{N}}$; if they are determined by finite sequences with specified bits on coordinates at most m and n, respectively, extend each to the coordinates in $[0, \max(m, n, j, k) + 1]$ by filling with all 0s in one and all 1s in the other. This handles $2^{\mathbb{N}}/\mathcal{F}$ because its relation is a subset of the tail relation. The argument for $2^{\mathbb{Z}} \times 2^{\mathbb{Z}}$ is similar, where one appends 0s to both ends of one finite string and 1s to the other. So Theorem 7 shows that the ambient set embeds into the set of classes for each of these relations.

In fact, embeddings as in Theorem 7 can be defined quite directly for many specific relations. Mycieski's original proof used the harmonic expansion of a real [**15**, **27**]). Here is an explicit embedding for the shift. Define $f: 2^{\mathbb{N}} \to 2^{\mathbb{Z}}/\text{shift}$ by $f(abc\ldots) = \ldots ccbbabbcc\ldots$, an equivalence class. Then a is the center of a unique constant block of odd length (the two constant sequences are easily handled) and this locates the origin and allows the recovery of $abc\ldots$ from its image. Therefore f is one-one, proving $|2^{\mathbb{Z}}| = |2^{\mathbb{N}}| \leq |2^{\mathbb{Z}}/\text{shift}|$.

For the second half of the division paradox, one can use the zero-one law in a way that avoids the use of a map from the space to itself (such as $x \mapsto -x$ in Theorem 3). Invariant sets are simply unions of some equivalence classes. The key point is that any invariant set for $2^{\mathbb{N}}/\mathcal{F}$, $2^{\mathbb{N}}/\text{tail}$, or $2^{\mathbb{Z}}/\text{shift}$ obeys the zero-one law: an invariant set having the Baire property is meager or comeager and an invariant measurable set has measure 0 or 1. For the measure case this is the same as saying that λ is *ergodic* with respect to the relation. The results for the tail relation follow from the same for $2^{\mathbb{N}}/\mathcal{F}$; so there are four cases to consider: $2^{\mathbb{N}}/\mathcal{F}$ and either meager or measure-zero sets and $2^{\mathbb{Z}}/\text{shift}$ and either meager or measure-zero sets. We present the details only for the last case.

Assume A is a shift-invariant measurable subset of $2^{\mathbb{Z}}$, with $\lambda(A) = \alpha$ and fix $\epsilon > 0$. Choose finitely many basic open sets $\hat{s}_i$ so that, with $E = \bigcup \hat{s}_i$, we have $A \subseteq E$ and $\lambda(E \setminus A) < \epsilon$ (this uses outer measure, as in the proof in Section 2). Let n be larger than the largest coordinate used in any s_i. Then the basic open sets that occur in $F = \sigma^n(E)$ have support disjoint from the basic sets $\hat{s}_i$; this yields that $\lambda(E \cap F) = \lambda(E)\,\lambda(F) = \lambda(E)^2$. Because $A \subseteq E$, we have $A = \sigma^n A \subseteq \sigma^n E = F$, so $A \subseteq E \cap F$ and the measure difference of these two sets is at most ϵ. Now,

$$\left|\alpha - \alpha^2\right| \le |\alpha - \lambda(E \cap F)| + \left|\lambda(E - \alpha^2\right| < \epsilon + \left|\alpha^2 - \lambda(E)^2\right| = \epsilon + (\lambda(E) + \alpha)\cdot|\lambda(E) - \alpha| \le 3\,\epsilon.$$

This proves $\alpha - \alpha^2 = 0$, so α is 0 or 1.

Instead of dealing with the specific details of measure 0 and meager sets, we can be more general. Let $\mathcal{I} \subseteq \mathcal{P}(X)$ be any countably complete ideal.

Definition. A set $A \subseteq X$ is *almost Borel* (w.r.t. $\mathcal{I}$) if for some Borel set B, $A \triangle B \in \mathcal{I}$; $A \subseteq X$ is *invariant* if A is closed under $\sim$. The relation and ideal satisfy the *zero-one law* if whenever A is an invariant set that is almost Borel, either $A \in \mathcal{I}$ or $X \setminus A \in \mathcal{I}$.

Now we can formulate a general result relating the zero-one law to the nonexistence of a certain injection.

Theorem 8 (ZF). *Suppose $\sim$ is a relation on X and $\mathcal{I} \subseteq \mathcal{P}(X)$* is an ideal so that: $X \notin \mathcal{I}$, each equivalence class is in $\mathcal{I}$, each invariant set is almost Borel, and the relation and ideal satisfy the zero-one law. *Then* $|X/\!\sim| \not\le |X|$.

Proof. Define a filter U on $X/\!\sim$ by placing A in U if $X \setminus \bigcup A$ is in $\mathcal{I}$; then, because $\bigcup A$ is invariant and therefore almost Borel, the zero-one law implies that U is an ultrafilter. An injection $f : X/\!\sim \to X$ would, by inverse image, transfer U to an ultrafilter on X and because $|X| = \left|2^{\mathbb{N}}\right|$, this gives an ultrafilter V on $2^{\mathbb{N}}$. For each k, split $2^{\mathbb{N}}$ into the set of sequences having a 0 in the kth position and the ones having a 1 there. One of these, call it A_k, is in V; then $\bigcap A_k$ is a singleton $\{s\}$ in V. But because $f^{-1}(\{s\})$ is either $\emptyset$ or a single equivalence class, it does not lie in U, contradiction. ■

Recall that any Lebesgue measurable set differs from a Borel set by a measure-zero set [**16**, §3] and by definition, a set with the Baire property differs from a Borel set by a meager set. Therefore one can let $\mathcal{I}$ be the measure-zero sets or the meager sets, depending on whether one is assuming LM or PB. If we then assume DC, the ideal $\mathcal{I}$ will be countably complete and Theorem 8 applies. In this way, Theorems 7 and 8 give the division paradox for $\mathbb{R}/\mathbb{Q}$, the finite set and tail relations on $2^{\mathbb{N}}$, and the Bernoulli shift on $2^{\mathbb{Z}}$, in the form given in Corollary 4.

We can eliminate the use of DC by a slightly different argument, one based on the proof of Theorem 3. We use $[x]$ for the equivalence class of x.

Theorem 9 (ZF). *Suppose the relation $\sim$ and ideal $\mathcal{I}$ satisfy the zero-one law and there is a function $\rho : X \to X$ such that*

1. $x \sim y$ if and only if $\rho(x) \sim \rho(y)$;
2. $\rho(\rho(x)) = x$ *for all* x.
3. *if* $A \in \mathcal{I}$, *then* $\rho(A) \in \mathcal{I}$;
4. $\{x \in X : x \nsim \rho(x)\}$ *is almost Borel and is not in* $\mathcal{I}$;

Suppose C is a choice set for $\{\{\gamma, \rho(\gamma)\} : \gamma \in X/\!\sim$ *and* $\gamma \ne \rho(\gamma)\}$. *Then* $\bigcup C$ *is not almost Borel w.r.t.* $\mathcal{I}$.

Proof. Note first that, by (1) and (2), $\rho(\gamma)$ is an equivalence class when γ is. Suppose $A = \bigcup C$ is almost Borel. Let $B = \rho(A)$; then, by (2), B is the union of the classes that were not chosen in C and we have the disjoint union $X = A \cup B \cup D$, where $D = \{x \in X : x \sim \rho(x)\}$. Because D is almost Borel by (4), so is B. Because A and B are unions of classes, they are $\sim$-invariant. Therefore, by the zero-one law, $A \in \mathcal{I}$ or $B \in \mathcal{I}$ (disjointness implies that the complements of both cannot be in $\mathcal{I}$). By (2) and (3) this means that both are in $\mathcal{I}$, so $A \cup B \in \mathcal{I}$, contradicting $X \setminus D \notin \mathcal{I}$, which holds by (4). ■

A function ρ as in Theorem 9 exists for our four examples, with either the meager or measure-zero ideal. For $\mathbb{R}$ use $x \mapsto -x$, while $s \mapsto 1 - s$ works for $2^{\mathbb{N}}/\mathcal{F}$ and the tail relation. For the shift relation, define ρ to be the reflection: $\rho(s)_n = s_{-n}$. Then (1–3) are clear. For (4), we have $\{s : s \sim \rho(s)\} = \bigcup_{k \in \mathbb{Z}} N_k$, where $N_k = \{s : s \text{ is a } k\text{-shift of } \rho(s)\}$. Then N_0 is the set of palindromic sequences with a single center element at the origin; N_1 is the set of palindromes with a double center, the rightmost of which is at the origin, and, for $k \ge 0$, N_{2k} (resp., N_{2k+1}) is the k-shift of N_0 (resp., N_1); the negative case is similar. Then each N_k is nowhere dense and has measure 0, and the same is true of their union. Moreover the proof does not use countable additivity and so works in ZF.

Here are the details in the measure case. We can cover N_0 by $[000] \bigcup [010] \bigcup [101] \bigcup [111]$, where $[abc]$ is the basic set consisting of all sequences extending abc, with the origin at b. This union has measure $4 \cdot 2^{-3} = 1/2$. Similarly

$$N_0 \subseteq [00000] \bigcup [00100] \bigcup [01010] \bigcup [01110] \bigcup [10001] \bigcup [10101] \bigcup [11011] \bigcup [11111]$$

a set of measure $8 \cdot 2^{-5} = 1/4$. Continuing in this way shows that N_0's measure can be made arbitrarily small. Similar coverings work for each N_i. Thus, for any $\epsilon > 0$, we can explicitly prove $\lambda(N_k) \leq \epsilon / 2^{k+1}$, which gives $\lambda(\bigcup N_k) < \epsilon$, as claimed.

Corollary 10 (ZF). *Under* LM *or* PB*, there is a division paradox in* $\mathbb{R}/\mathbb{Q}$, $2^{\mathbb{N}}/\mathcal{F}$, $2^{\mathbb{N}}/$*tail, and* $2^{\mathbb{Z}}/$*shift. That is, in each case* $|X| < |X/\sim|$.

Proof. Assume PB and let $\mathcal{I}$ be the meager ideal. Theorem 7 gives the injection of X into $X/\sim$. Let ρ be as given before the corollary. If there were an injection $f: X/\sim \to X$, then there would be a choice set C as in Theorem 9 (choose γ if $f(\gamma)$ is smaller than $f(\rho(\gamma))$ in the natural linear ordering of X, otherwise choose $\rho(\gamma)$). Now apply Theorem 9 to conclude that C does not differ from a Borel set by a meager set, in contradiction to all sets having the Baire property. The proof under LM is the same, using measure-zero sets. ■

We conclude with some connections to modern descriptive set theory. One can show [**2**, Theorem 7.1] in ZF that $\mathbb{R}/\mathbb{Q}$, $2^{\mathbb{N}}/\mathcal{F}$, $2^{\mathbb{N}}/$tail, and $2^{\mathbb{Z}}/$shift all have the same cardinality. This means that the division paradox in one of these yields the paradox for all of them. This was how the original paradox in $\mathbb{R}/\mathbb{Q}$ was derived by Mycielski: he started with $2^{\mathbb{N}}/\mathcal{F}$. But if one introduces the proper notion of isomorphism, then these four structures collapse to three.

The proper definition is *Borel isomorphism*: the existence of a Borel bijection $f: X \to Y$ so that $x \sim x'$ if and only if $f(x) \sim f(x')$ (see [**2**, Section 9]). One then has that the three relations based on the power sets are not Borel isomorphic, while $\mathbb{R}/\mathbb{Q}$ is Borel isomorphic to one of them. It is not immediately obvious which of the power set quotients will be isomorphic to $\mathbb{R}/\mathbb{Q}$!

It turns out that $\mathbb{R}/\mathbb{Q}$ is Borel isomorphic to $2^{\mathbb{N}}/$tail. The reason is that these two relations have a property that the others do not. A relation $\sim$ on X is *paradoxical* if X contains disjoint A_1 and A_2 and there are Borel bijections $f_i: X \to A_i$ so that, for all x, $x \sim f_i(x)$. Then the tail relation on $2^{\mathbb{N}}$ is paradoxical via the functions f_i that prepend i to each sequence. And $\mathbb{R}/\mathbb{Q}$ is paradoxical via $\mathbb{R} = (-\infty, 0] \bigcup (0, \infty)$; $\mathbb{R}$ maps to $(0, \infty)$ (the negative case is similar) by sending $(n, n+1]$ to $(m, m+1]$ ($n \in \mathbb{Z}$, $m \in \mathbb{N}$), using alternation to fit them all in. The other two relations are not paradoxical in this sense, and so cannot be Borel isomorphic to $\mathbb{R}/\mathbb{Q}$.

6. OPEN QUESTIONS. The Banach–Tarski paradox violates the intuition one has from physical reality and LM eliminates the paradox. But the connection with topology is more subtle. Dougherty and Foreman (see [**26**, §11.2]) proved that, in ZFC, one can derive the Banach–Tarski paradox with the pieces all having the property of Baire. But it is not known whether PB eliminates the classic version of the paradox; it seems reasonable to conjecture that the answer to Question 1 is NO.

Question 1. Is the negation of the Banach–Tarski paradox a theorem of ZF + DC + PB?

Let GM (for *general measure*) be: For each n, there is a countably additive, isometry-invariant measure on $\mathcal{P}(\mathbb{R}^n)$ that assigns measure 1 to the unit cube. Then DC + LM implies GM. But GM eliminates the Banach–Tarski paradox and has the advantage that its consistency does not require a large cardinal assumption (see [**26**, §15.1]). Because the zero-one law uses outer measure, it is not clear that the proof of Theorem 3 can be modified to work under GM and so we have the following question.

Question 2. Is the division paradox a theorem of ZF + DC + GM?

To understand how the division paradox relates to more general statements, we recall two classical principles (see [**1**] for the history of these).

Definition. The *partition principle* PP is: If Y is a family of disjoint nonempty subsets of X, then $|Y| \leq |X|$. The w*eak partition principle* WPP is: If Y is a family of disjoint nonempty subsets of X, then $|X| \not< |Y|$.

If WPP holds then one cannot have the division paradox for $\mathbb{R}$ and $\mathbb{R}/\mathbb{Q}$, or for any other sets. Easy implications are AC ⇒ PP ⇒ WPP. More subtle are PP ⇒ DC and PP ⇒ AC for well-ordered families (A. Pelc; see [**12**, p. 10]).

Note that a violation of PP is less problematic than a violation of WPP. If we did not have Theorem 2 and knew only that, under LM, $|\mathbb{R}/\mathbb{Q}| \not\leq |\mathbb{R}|$ (i.e., $\mathbb{R}$ and $\mathbb{R}/\mathbb{Q}$ violated PP, as opposed to WPP), it would not be all that disturbing as it says only that a certain injection does not exist. The interesting point is Corollary 4, which states that $\mathbb{R}$ and $\mathbb{R}/\mathbb{Q}$ are comparable, but the comparison goes the *wrong* way: the set of Vitali classes of real numbers has more elements than the set of reals!

These partition principles lead to two fascinating open questions.

Question 3. (a) Does PP imply AC? (b) Does WPP imply AC?

An affirmative answer to (b) would be of some importance. For then we could abandon AC as an axiom and work with ZF + WPP, in the knowledge that it is no different than ZFC. But WPP feels more fundamentally obvious than the axiom of choice. Of course, absent any proofs, this is only speculation and it would not be terribly surprising if WPP was strictly weaker than AC.

7. CONCLUSION. We believe that the division paradox is an obviously false statement because of how seriously it undermines our intuition about how sets work—even more so than the Banach–Tarski paradox. By Corollary 4, this implies that ZF + LM is an untenable theory. On the other hand, the ball-duplication paradox really presents no serious problems. The sets that arise are not connected to physical reality, and a century of development has shown that there are no serious consequences of living with nonmeasurable sets.

Of course one can take the opposite view and consider the division paradox a small price to pay in order to be rid of nonmeasurable sets (see [**14**]); but this price strikes us as being excessively high due to its counterintuitive nature. A more extreme view, presented by Feferman [**3**], is to view the duplication paradox as being so severe as to call for the banishment of the unrestricted use of sets.

It makes little sense to say that either the Banach–Tarski paradox or the negation of the division paradox is true in physical reality. Instead we are trying to understand which formal axiom system best describes mathematics as it is practiced. Mathematics appears to work well under a system that combines the best aspects of Platonism (mathematics describes a real world) and formalism (examine proofs under diverse axiom systems). Our perceived physical world certainly affects how we think of much of mathematics, but formalism combines the precision of proofs with the possibility of imaginatively and profitably examining a variety of axiom systems in realms beyond physical reality. Despite the Banach–Tarski paradox, the axiom system ZFC has shown itself to be up to the task of serving as a solid foundation for mathematics.

8. NOTES.

1. LM negates the Banach–Tarski paradox. Because $1 + 1 \neq 1$, it suffices to show that all subsets of $\mathbb{R}^3$ are Lebesgue measurable. Cantor's classic digit-mixing bijection from $[0,\infty)$ to $[0,\infty)^3$ uses the positions congruent to i (mod 3) ($i = 0, 1, 2$) to split the decimal digits into three infinite strings. This function preserves the measure of intervals (in dimensions 1 and 3) and hence preserves outer measure. Therefore all subsets of the first octant are measurable and the same then holds for all octants and, by finite additivity, for all subsets of $\mathbb{R}^3$. This is also a consequence of more general characterization theorems; see [**11**, Theorem 17.41] or [**8**, Appendix A].

2. Measure and large cardinals. While there are many similarities between the concepts of Lebesgue measure and meager sets, there is one remarkable difference. It is that LM is stronger in that its consistency requires the consistency of IC, the assertion that an inaccessible cardinal exists. To summarize much seminal and difficult work by Solovay, Shelah, and others, let Con(T) mean that the theory T is consistent. Then

- ZF + IC $\Rightarrow$ Con(ZF); hence by Gödel's second incompleteness theorem we cannot derive Con(ZF + IC) from Con(ZF).
- Con(ZF) $\Leftrightarrow$ Con(ZF + DC + PB).
- Con(ZF + IC) $\Leftrightarrow$ Con(ZF + DC + LM).

In short, the consistency strength of LM is strictly greater than that of PB (but see the discussion of GM in Section 6 for how to avoid inaccessibles when negating the Banach–Tarski paradox).

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