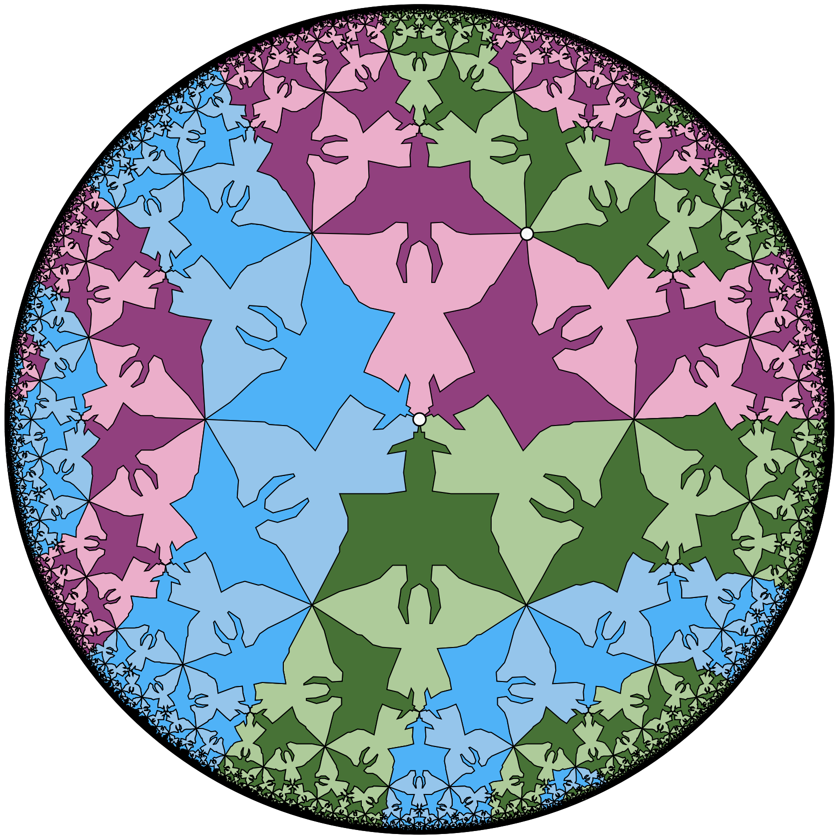

About the Cover:

Escher, Fricke, Hausdorff, and Klein: A Paradoxical Connection

Curtis D. Bennett (curtis.bennett@csulb.edu) and Stan Wagon (wagon@macalester.edu)

Affiliations: California State University, Long Beach, CA; Macalester College, St. Paul, MN

The cover image shows how ideas of Klein, Fricke, Hausdorff, and Escher can be used to partition the hyperbolic plane $\mathbb{H}^2$ in a way that provides a concrete view of a paradoxical decomposition. The key to the famous Banach–Tarski paradox (1924; see [2]) is a decomposition found by Felix Hausdorff (1914) in the group $G = \mathbb{Z}_2 * \mathbb{Z}_3$. This group consists of all words in σ and τ where $\sigma^2 = \tau^3 =$ identity; that is: $G = \langle \sigma, \tau \mid \sigma^2 = \tau^3 = e \rangle$. The group, which is the free product of $\mathbb{Z}_2$ and $\mathbb{Z}_3$, arises prominently in a famous tiling of $\mathbb{H}^2$, found by Klein and Fricke in 1890 (Figure 2). Hausdorff saw how to partition G into $A \cup B \cup C$ such that $A \sim B \sim C$ and also $A \sim B \cup C$, where $\sim$ denotes congruence in the group (Figure 1). Thus A can be viewed as being a third of G, and also half of G. He then found that the group could be represented as rotations of a sphere. The three sets can then, with the help of the Axiom of Choice, be lifted to the sphere, less D, a set of countably many exceptional points (D consists of the points on the axes of the rotations that lie on the sphere). That construction leads to three subsets of the sphere minus D that are paradoxical (each is a third, and also a half). Banach and Tarski worked out how to absorb D into the construction, allowing them to define pairwise disjoint subsets of the sphere that be rearranged to form two copies of the sphere: the Banach–Tarski Paradox.

Hausdorff's three sets can be defined as follows, where, for $\alpha \in \{\sigma, \tau, \tau^2\}$, W_α consists of all words starting on the left with α. Set $A = W_\sigma$, $B = W_\tau$, $C = W_{\tau^2}$. Then, ignoring e, we have the desired identities $\sigma(A) = B \cup C$, $\tau A = B$, and $\tau^2 A = C$. To include e, let $\beta = \tau^2 \sigma$, place e into A, and move the words $\beta, \tau\beta, \tau^2 \beta, \beta^2, \tau\beta^2, \tau^2 \beta^2, \beta^3, \tau\beta^3, \tau^2 \beta^3, \beta^4, \ldots$ in order, into $A, B, C, A, B, C, A, B, C, A, \ldots$. This construction absorbs the identity and yields the desired congruences (Figure 1; note that $\tau\beta = \sigma$). This is similar to how the fixed point set D was handled by Banach and Tarski to get their Euclidean paradox.

A	B	C
e	τ	$\tau\tau$
β	$\tau\beta$	$\tau\tau\beta$
$\sigma\tau$	$\tau\sigma\tau$	$\tau\tau\sigma\tau$
$\sigma\tau\sigma$	$\tau\sigma\tau\sigma$	$\tau\tau\sigma\tau\sigma$
$\sigma\tau\tau$	$\tau\sigma\tau\tau$	$\tau\tau\sigma\tau\tau$
$\sigma\tau\sigma\tau$	$\tau\sigma\tau\sigma\tau$	$\tau\tau\sigma\tau\sigma\tau$
β^2	$\tau\beta^2$	$\tau\tau\beta^2$
$\sigma\tau\sigma\tau\sigma$	$\tau\sigma\tau\sigma\tau\sigma$	$\tau\tau\sigma\tau\sigma\tau\sigma$
$\sigma\tau\sigma\tau\tau$	$\tau\sigma\tau\sigma\tau\tau$	$\tau\tau\sigma\tau\sigma\tau\tau$
$\sigma\tau\tau\sigma\tau$	$\tau\sigma\tau\tau\sigma\tau$	$\tau\tau\sigma\tau\tau\sigma\tau$
$\sigma\tau\sigma\tau\sigma\tau$	$\tau\sigma\tau\sigma\tau\sigma\tau$	$\tau\tau\sigma\tau\sigma\tau\sigma\tau$
$\sigma\tau\sigma\tau\tau\sigma$	$\tau\sigma\tau\sigma\tau\tau\sigma$	$\tau\tau\sigma\tau\sigma\tau\tau\sigma$
$\sigma\tau\tau\sigma\tau\sigma$	$\tau\sigma\tau\tau\sigma\tau\sigma$	$\tau\tau\sigma\tau\tau\sigma\tau\sigma$
$\sigma\tau\tau\sigma\tau\tau$	$\tau\sigma\tau\tau\sigma\tau\tau$	$\tau\tau\sigma\tau\tau\sigma\tau\tau$
$\sigma\tau\sigma\tau\sigma\tau\sigma$	$\tau\sigma\tau\sigma\tau\sigma\tau\sigma$	$\tau\tau\sigma\tau\sigma\tau\sigma\tau\sigma$
$\sigma\tau\sigma\tau\sigma\tau\tau$	$\tau\sigma\tau\sigma\tau\sigma\tau\tau$	$\tau\tau\sigma\tau\sigma\tau\sigma\tau\tau$
$\sigma\tau\sigma\tau\tau\sigma\tau$	$\tau\sigma\tau\sigma\tau\tau\sigma\tau$	$\tau\tau\sigma\tau\sigma\tau\tau\sigma\tau$
$\sigma\tau\tau\sigma\tau\sigma\tau$	$\tau\sigma\tau\tau\sigma\tau\sigma\tau$	$\tau\tau\sigma\tau\tau\sigma\tau\sigma\tau$
β^3	$\tau\beta^3$	$\tau\tau\beta^3$
$\sigma\tau\sigma\tau\sigma\tau\sigma\tau$	$\tau\sigma\tau\sigma\tau\sigma\tau\sigma\tau$	$\tau\tau\sigma\tau\sigma\tau\sigma\tau\sigma\tau$
$\sigma\tau\sigma\tau\sigma\tau\tau\sigma$	$\tau\sigma\tau\sigma\tau\sigma\tau\tau\sigma$	$\tau\tau\sigma\tau\sigma\tau\sigma\tau\tau\sigma$
$\sigma\tau\sigma\tau\tau\sigma\tau\sigma$	$\tau\sigma\tau\sigma\tau\tau\sigma\tau\sigma$	$\tau\tau\sigma\tau\sigma\tau\tau\sigma\tau\sigma$

Figure 1. The three sets of the Hausdorff Paradox; here $\beta = \tau^2 \sigma$ and the red entries form the infinite sequence that absorbs the identity. Note that $\tau\beta = \sigma$.

The Klein–Fricke tiling arises by repeated reflection starting from the triangle labeled e in Figure 2, which uses the upper-half-plane model; each triangle has a vertex at ∞ and the corresponding group is the triangle group $T_{3,3,\infty}$. Each tile can be labeled with an element of G, and Mycielski and Wagon (1984) had the idea of coloring the tiles according to which of A, B, C contains the label. The congruences in the group then become congruences in $\mathbb{H}^2$, leading to a geometric visualization of the Hausdorff paradox [2,

Figure 4.5]; for an animation see [3].

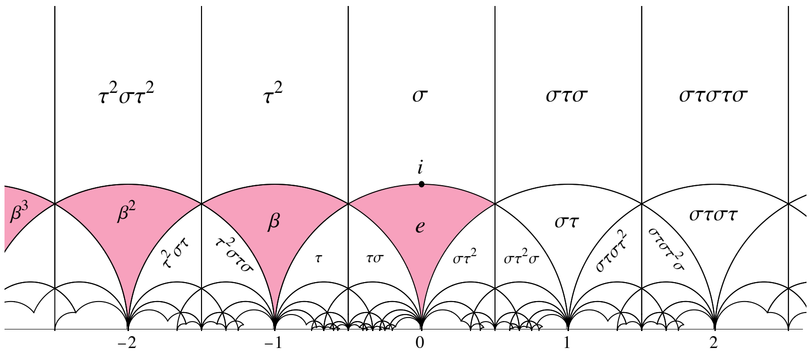

Figure 2. The Klein–Fricke tiling of the hyperbolic plane, viewed in the upper half-plane model. Multiplication by β corresponds to subtracting 1, so the absorption of e is handled by the tail of large triangles heading west.

The question now arises whether a similar construction exists for one of the artistic tilings of $\mathbb{H}^2$ used by M. C. Escher. Bennett, working with the group

$$H = \langle \sigma, \tau, \rho \mid \sigma^2 = \tau^2 = \rho^2 = (\sigma\,\tau)^3 = (\tau\,\rho)^4 = (\rho\,\sigma)^4 = e \rangle,$$

succeeded in this idea [1]. This group is also known as the triangle group $T_{3,4,4}$. The tiling of $\mathbb{H}^2$ corresponding to this group is shown in Figure 3, where the Poincaré disk model is used. As before, each triangle corresponds to an element of H and so a decomposition of H into three paradoxical sets leads to the same for $\mathbb{H}^2$. But now the tiling matches Escher's and so each tile can be viewed as either an angel or devil. The three sets arise by first taking the subgroup G' of H generated by $(\tau\rho)^2$ and $\sigma\tau$. Then G' is isomorphic to the group G used by Hausdorff and the paradoxical subsets of G can be used to obtain a similar paradox in H.

The division into angels and devils is unimportant for the paradoxical coloring. In the images, angels are elements of H with odd length (i.e., in the free product they correspond to a product of an odd number of generators), while the devils have even length. Unlike the tiling in Figure 1, this tiling has only finite triangles.

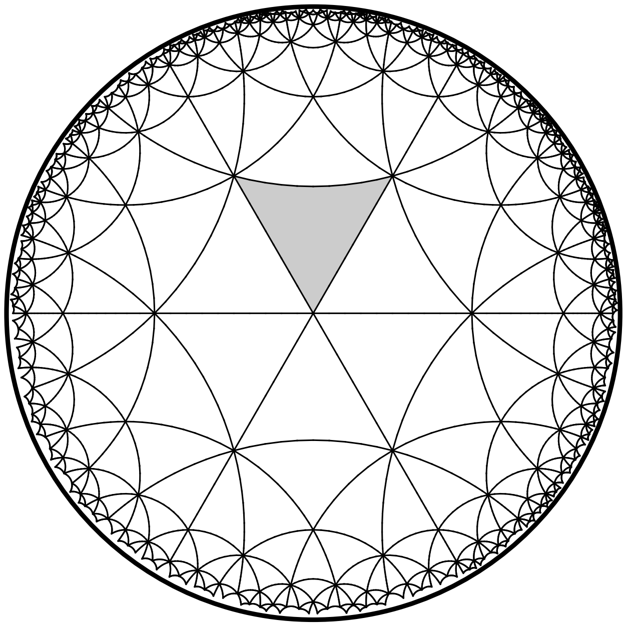

Figure 3. The hyperbolic tiling corresponding to the group H; this underlies Escher's *Angels and Devils* woodcut.

The cover image shows the three sets in magenta, green, and cyan, with a viewpoint is directly above the origin; it is clear that the three sets are congruent by rotations. When the viewpoint is moved to $\frac{\sqrt{3}-1}{\sqrt{2}}\left(\frac{1}{2}+\sqrt{\frac{3}{2}}\, i\right)$ (the white disk at upper right on the cover), and one combines the green and cyan regions into a single set, then it is evident that the magenta set is congruent to its complement (Figure 4). Thus the magenta set is both one third and one half of the hyperbolic plane. This paradox is not at all problematic because the standard measure of measurable sets in $\mathbb{H}^2$ assigns measure ∞ to the entire plane.

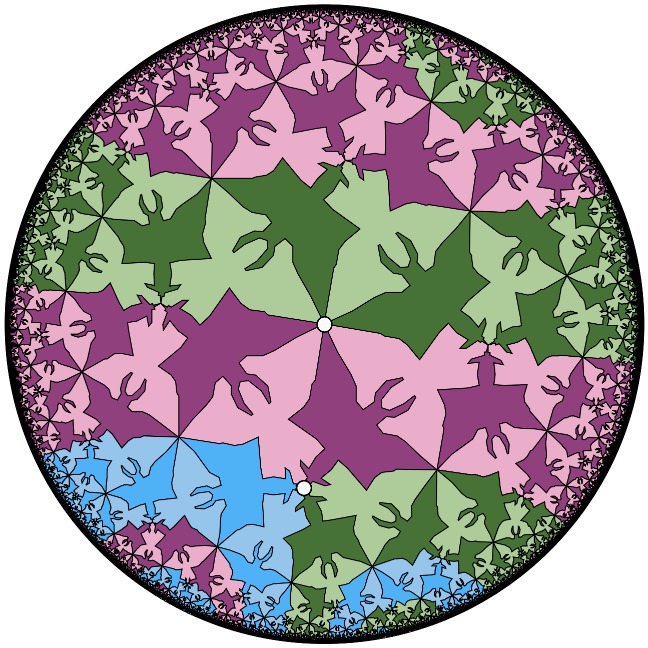

Figure 4. The cover image viewed from above the point $\frac{\sqrt{3}-1}{\sqrt{2}}\left(\frac{1}{2}+\sqrt{\frac{3}{2}}\,i\right)$. In this view it is clear that the magenta set is a rotation of its complement.

Paradoxes generally mean that certain measures cannot exist. A *Mycielski measure* on $\mathbb{R}^n$ is a finitely additive, isometry invariant measure on the Lebesgue measurable sets having total measure 1 (see [2], p. 261). These measures exist for all $\mathbb{R}^n$, so a paradox similar to those discussed here cannot exist in Euclidean space. But the constructive paradox in $\mathbb{H}^2$ means that a hyperbolic Mycielski measure cannot exist in dimension 2, or higher. This highlights an interesting distinction between hyperbolic and Euclidean space.

The Banach–Tarski Paradox is considered one of the most surprising and bizarre theorems of mathematics. Constructive versions of the paradox can shed light on what underlies the construction, and it is remarkable that classic ideas of hyperbolic geometry can combine with Escher's artistic work in the hyperbolic plane to bring the paradox to life.

REFERENCES

1. C. D. Bennett, 2000. A paradoxical view of Escher's Angels and Devils, *Mathematical Intelligencer*, 22:3, 39–46.

2. G. Tomkowicz and S. Wagon, 2016. *The Banach–Tarski Paradox*, 2nd ed., Cambridge Univ. Pr.

3. S. Wagon, 2006. The Banach–Tarski Paradox, Wolfram Demonstrations Project. https://demonstrations.wolfram.com/TheBanachTarskiParadox/