

**Theorem.** (Gregory Galperin, Stephen Butler, Ron Graham, and Richard Stong) Consider this transformation of a positive integer. Write it in binary, insert any nonzero number of plus signs into the bit string at any places you choose (never two in a row), and perform all the indicated additions. For example,

$$98765 = 11000011010011111_2 \rightarrow 1100 + 001 + 1010 + 011 + 11 + 1$$

which is  $12 + 1 + 10 + 3 + 3 + 1$ , or 30. This transformation reduces the number and so repetition will eventually reach 1.

Show that one can always reach 1 using only two transformations.

Proof (Feb. 2026, by Peter Winkler). Use induction to prove that a power of 2 can always be reached in one step. Let  $w(L)$  be the number of 1s in the bit-string  $L$ . Let  $k$  denote the smallest integer such that  $w(L) \leq 2^k$ . Let the *basic form* of a bit-string be the expression that results by inserting + in every possible location. Preliminary observations:

1. If  $w(L) = 2^k$ , the basic form has sum  $2^k$ .
2. If  $w(L) = 3$ , we can get sum 4. Proof. Use the basic form and transform a single  $1 + x$  to  $1x$ . This adds 1 to the sum, bringing it from 3 to 4.
3. If  $w(L) = 4$ , we can get sum 8. Proof. One of the following cases must occur (only one of each merge is done):
  - $1 + 1 + x \rightarrow 11x$  adds 4;
  - $1 + 0 + x \rightarrow 10x$  and at least one  $1 + 0 \rightarrow 10$ , disjoint from  $1 + 0 + x$ , turns 3 to 7, adding 4;
  - $1 + 0 + 1 \rightarrow 101$  and at least one  $1 + 1 \rightarrow 11$ , disjoint from  $1 + 0 + 1$ , turns 4 to 8, adding 4.
4. If  $w(L) = 5$ , we can get 8 in all cases except 11111, which yields 16. Proof.  $11111 \rightarrow 1 + 1111$ ;  $111110 \rightarrow 11 + 11 + 10$ ; otherwise the basic form contains  $1 + 0 + x$ , which can be merged to

$10x$ , adding 3.

We will use a two-pronged induction hypothesis. A: If  $2^{k-1} + 1 \leq w(L) \leq 2^k$ , we can attain sum  $2^k$  in one transformation. And B: If  $w(L) = 2^k$ , we can attain  $2^{k+1}$  in one transformation.

These assertions fail when  $w = 1, 2$ , or  $5$ , so we will prove that they hold whenever  $w \geq 6$ , which solves the problem. Note the key fact (3) that  $w = 4$  can reach the sum of 8. We will in all cases split  $L$  into parts of weight  $u$  and  $v$ , where  $u + v = w(L)$ , and apply known or inductive cases to the parts to get the same two sums, which add to the desired power of 2.

If  $w = 6$  or  $7$ , take  $u = w - 3$  and  $v = 3$ , getting 4 for each piece to confirm (A).

If  $w = 8$ , take  $u = v = 4$ , getting 4 for each piece to confirm (A), and 8 from each piece to confirm (B).

If  $w = 9$ , specify first the special case  $n = 511 = 111111111_2 \rightarrow 111 + 11 + 11 + 11$ , which is 16 and confirms (A). Otherwise let  $(u, v)$  be either  $(5, 4)$  or  $(4, 5)$ . There is a 0 somewhere so we can choose  $u$  such that the corresponding piece is not the dreaded 1111. Then, by (3) and (4), we get 8 from each piece and the total of 16 confirms (A).

If  $w = 10$  or  $11$ , use  $u = w - 4$  and  $v = 4$ , using (A) to get 8 for  $u$  and (B) to get 8 for  $v$ , summing to 16 to confirm (A).

For  $w \geq 12$ , use  $u$  and  $v$  with  $u - v = 0$  or  $1$ . There are three cases. If  $w = 2^k$ , let  $u = v = 2^{k-1}$  and use the inductive hypothesis (B) to attain the sum  $2^k$  for each piece, yielding  $2^{k+1}$  when they are combined. This establishes (B). If  $2^{k-1} + 2 \leq w(L) \leq 2^k$  then  $2^{k-2} + 1 \leq u, v \leq 2^{k-1}$  and we can apply (A) to both  $u$  and  $v$  (note that  $u, v \geq 6$ ) to get sums of  $2^{k-1}$ , which combine to give  $2^k$ , proving (A). Finally, if  $w(L) = 2^{k-1} + 1$ , then  $u = 2^{k-2} + 1$  and  $v = 2^{k-2}$ .

Use (A) to get  $2^{k-1}$  for the first piece and (B) to get  $2^{k-1}$  for the second piece; these combine to give  $2^k$ , proving (A).  $\square$